§14.3

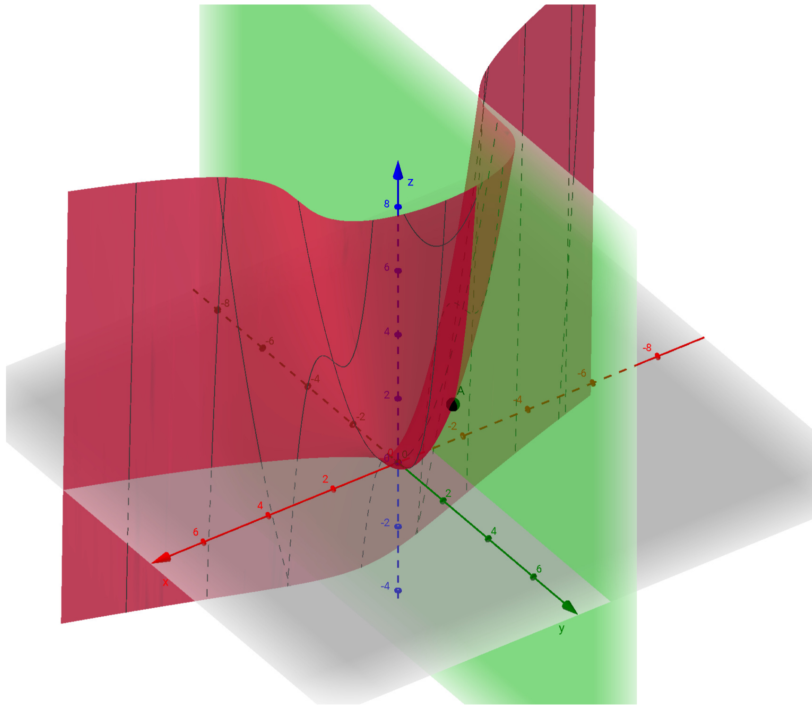
62. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in the **a.** plane $x = -1$ **b.** plane $y = 1$.

(a) $x = -1$,

$f_y(x, y) = 3y^2$

$f_y(-1, 1) = 3$

$\therefore$ slope $= 3$

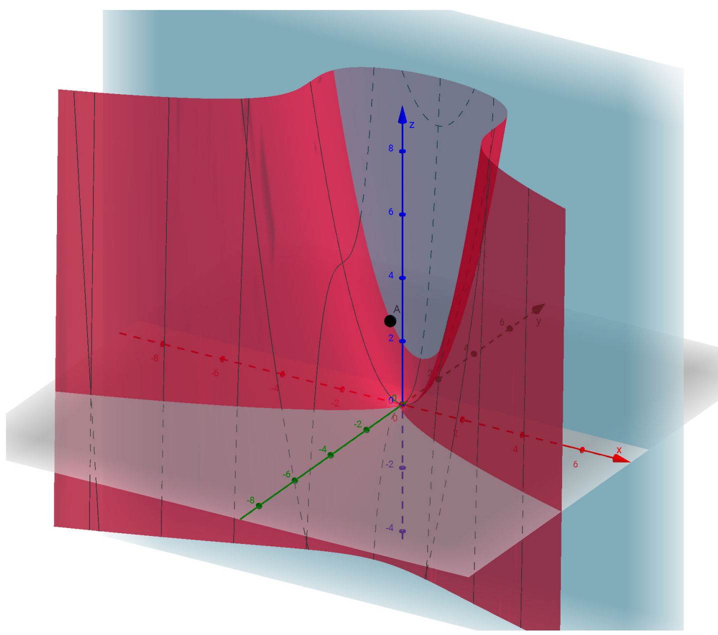


(b) $y = 1$,

$f_x(x, y) = 2x$

$f_x(-1, 1) = -2$

$\therefore$ slope $= -2$



66. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

$$xz + y \ln x - x^2 + 4 = 0$$

$$\frac{\partial}{\partial z}\left(xz + y \ln x - x^2 + 4\right) = \frac{\partial}{\partial z}(0)$$

$$\frac{\partial x}{\partial z} z + x \frac{\partial z}{\partial z} + \frac{\partial y}{\partial z} \ln x + \frac{y}{x}\frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} = 0$$

y, z are independent variables $\Rightarrow \frac{\partial y}{\partial z} = 0$

Then
$$\frac{\partial x}{\partial z}\left(z + \frac{y}{x} - 2x\right) = -x.$$

At $(1, -1, -3)$,

$$\frac{\partial x}{\partial z}(-3 - 1 - 2) = -1$$

$$\frac{\partial x}{\partial z} = \frac{1}{6}.$$

75. $f(x, y) = e^{-2y} \cos 2x$

Show f satisfy Laplace Equation.

$$\frac{\partial f}{\partial x} = e^{-2y}(-2\sin 2x)$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-2y}(-4\cos 2x)$$

$$\frac{\partial f}{\partial y} = -2e^{-2y}\cos 2x$$

$$\frac{\partial^2 f}{\partial y^2} = 4e^{-2y}\cos 2x$$

$$\therefore \ \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

81. $w = \sin(x + ct)$

Show w satisfy Wave Equation.

$$\frac{\partial w}{\partial x} = \cos(x + ct)$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x + ct)$$

$$\frac{\partial w}{\partial t} = c\cos(x + ct)$$

$$\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x + ct)$$

$$\therefore \quad \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

91. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$. (*Hint:* Use Theorem 4 and show that f is not continuous at $(0, 0)$.)

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h \cdot 0}{h^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{0}{h^3} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{0 \cdot h^2}{h^4} - 0}{h}$$

$$= \lim_{h \to 0} \frac{0}{h^5} = 0$$

If:

f is differentiable at $(0,0)$

$\Rightarrow$ f is continuous at $(0,0)$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0) \\ \text{on } x=y^2}} f(x,y) = 0$$

But $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ \text{on } x=y^2}} f(x,y)$

$$= \lim_{y\to 0} \frac{y^2 y^2}{y^4 + y^4}$$

$$= \frac{1}{2}.$$

$\therefore$ f is not differentiable at $(0,0)$

(Or showing f not differentiable by definition,

or showing $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.)

§14.5

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$

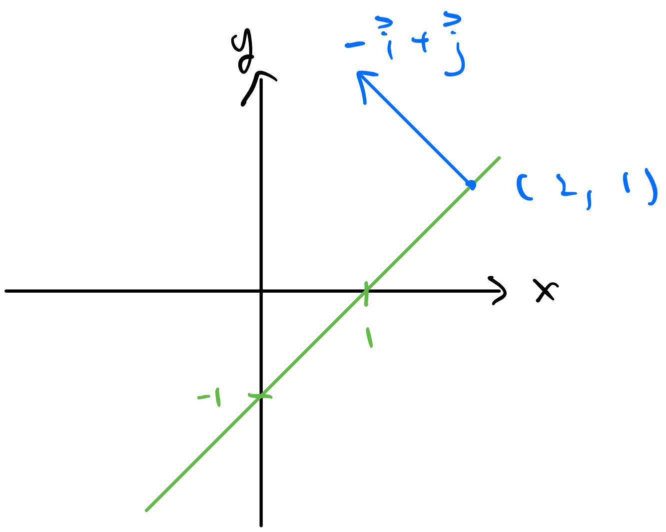
$$\frac{\partial f}{\partial x} = -1 \qquad \frac{\partial f}{\partial y} = 1$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore \nabla f(2, 1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\vec{i} + \vec{j}$$

$$f(2, 1) = -1$$

$\therefore$ The level set is $\{(x, y) \mid y - x = -1\}$



In Exercises 7–10, find ∇f at the given point.

7. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, \quad (1, 1, 1)$

$$\frac{\partial f}{\partial x} = 2x + \frac{z}{x} \qquad \left.\frac{\partial f}{\partial x}\right|_{(1,1,1)} = 3$$

$$\frac{\partial f}{\partial y} = 2y \qquad \left.\frac{\partial f}{\partial y}\right|_{(1,1,1)} = 2$$

$$\frac{\partial f}{\partial z} = -4z + \ln x \qquad \left.\frac{\partial f}{\partial z}\right|_{(1,1,1)} = -4$$

$$\therefore \nabla f(1,1,1) = \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_{(1,1,1)} \\ \left.\frac{\partial f}{\partial y}\right|_{(1,1,1)} \\ \left.\frac{\partial f}{\partial z}\right|_{(1,1,1)} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$= 3\vec{i} + 2\vec{j} - 4\vec{k}.$$

In Exercises 11–18, find the derivative of the function at P_0 in the direction of **u**.

13. $g(x, y) = \dfrac{x - y}{xy + 2}$, $P_0(1, -1)$, $\mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$

$$g_x(x,y) = \frac{(xy+2)(1) - (x-y)(y)}{(xy+2)^2}$$

$$g_x(1,-1) = 3$$

$$g_y(x,y) = \frac{(xy+2)(-1) - (x-y)(x)}{(xy+2)^2}$$

$$g_y(1,-1) = -3$$

$$\therefore \nabla g(1,-1) = 3\vec{i} - 3\vec{j}$$

Denote Direction of $\vec{u}$ by $\vec{v}$

$$\vec{v} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$= \frac{12\vec{i} + 5\vec{j}}{\sqrt{12^2 + 5^2}}$$

$$= \frac{12}{13}\vec{i} + \frac{5}{13}\vec{j}$$

$$\therefore D_{\vec{v}}\, g(1,-1) = \frac{21}{13}$$

In Exercises 19–24, find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

19. $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$

$$\nabla f = f_x \vec{i} + f_y \vec{j}$$

$$= (2x+y)\vec{i} + (x+2y)\vec{j}$$

$$\nabla f(-1,1) = -\vec{i} + \vec{j}$$

Direction of greatest increase $\vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$

$$= -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

Direction of greatest decrease $\vec{v} = -\dfrac{\nabla f}{\|\nabla f\|}$

$$= \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

$$D_{\vec{u}} f(-1,1) = \nabla f(-1,1) \cdot \vec{u}$$

$$= \sqrt{2}$$

$$D_{\vec{v}} f(-1,1) = \nabla f(-1,1) \cdot \vec{v}$$

$$= -\sqrt{2}.$$

29. Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f(1, -1)$ for which

a. $D_{\mathbf{u}} f(1, -1)$ is largest **b.** $D_{\mathbf{u}} f(1, -1)$ is smallest

c. $D_{\mathbf{u}} f(1, -1) = 0$ **d.** $D_{\mathbf{u}} f(1, -1) = 4$

e. $D_{\mathbf{u}} f(1, -1) = -3$

$$\nabla f = (2x - y)\vec{i} + (-x + 2y - 1)\vec{j}$$

$$\nabla f(1,-1) = 3\vec{i} - 4\vec{j}$$

(a) & (b):

Note $\|\vec{u}\| = 1$.

By Cauchy-Schwarz Inequality,

$$|D_{\vec{u}} f| = |\nabla f \cdot \vec{u}| \leq \|\nabla f\| \cdot \|\vec{u}\|$$

$$= \|\nabla f\| \text{ as } \vec{u} \text{ is unit.}$$

and equality holds iff $\nabla f = a\vec{u}$, $a \in \mathbb{R}$.

So,

$$-5 \leq D_{\vec{u}} f(1,-1) \leq 5$$

Need to find $a_1, a_2 \in \mathbb{R}$

s.t. $5 = D_{\vec{u}} f(1,-1)$ when $\vec{u} = a_1 \nabla f$

$-5 = D_{\vec{u}} f(1,-1)$ when $\vec{u} = a_2 \nabla f$

$$5 = D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \vec{u}$$
$$= \nabla f \cdot a_1 \nabla f$$
$$= a_1 5^2$$

$\therefore a_1 = \frac{1}{5}$

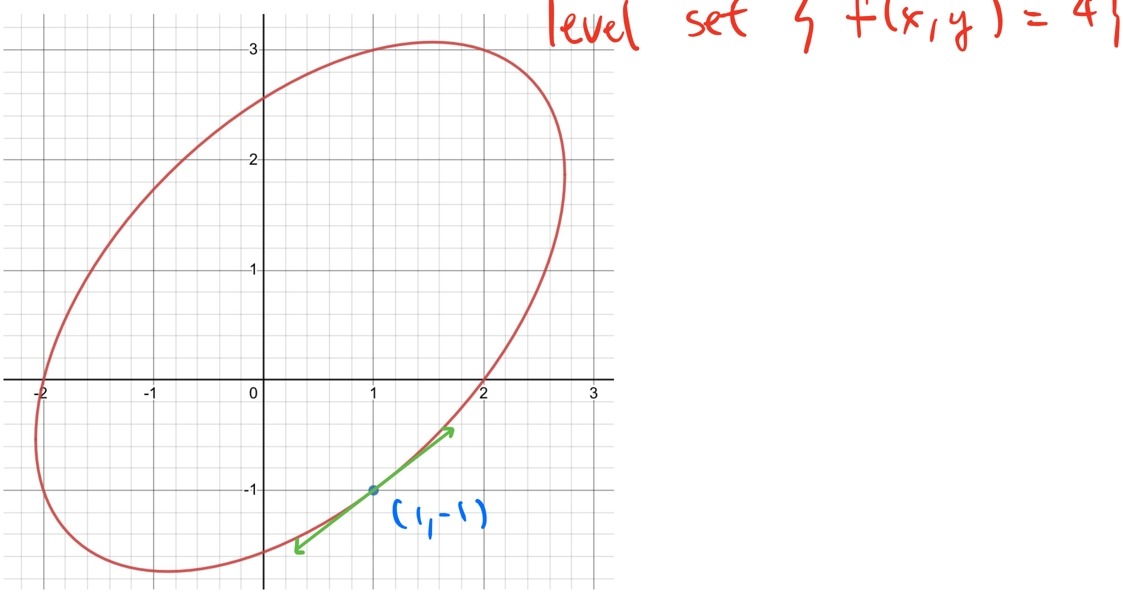
$$-5 = D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \vec{u}$$
$$= \nabla f \cdot a_2 \nabla f$$
$$= a_1 5^2$$

$\therefore a_2 = -\frac{1}{5}$

$\therefore$ (a) : $\vec{u} = \frac{1}{5} \nabla f(1,-1)$
$$= \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$

(b) : $\vec{u} = -\frac{1}{5} \nabla f(1,-1)$
$$= -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

(c). $f(1,-1) = 4$



Travelling along level set will not increase or decrease value of f.

So the vectors along tangent to the level set satisfy $D_{\vec{u}} f = 0$.

Tangent vectors can be found using implicit differentiation:

$$x^2 - xy + y^2 - y = 4$$

$$\frac{\partial}{\partial x}\left(x^2 - xy + y^2 - y\right) = \frac{\partial}{\partial x}(4)$$

$$2x - y - x\frac{\partial y}{\partial x} + 2y\frac{\partial y}{\partial x} - \frac{\partial y}{\partial x} = 0$$

$$\left.\frac{\partial y}{\partial x}\right|_{(1,-1)} = \frac{3}{4}$$

Then $\vec{v} = \vec{i} + \left.\frac{dy}{dx}\right|_{(1,-1)} \vec{j} = \vec{i} + \frac{3}{4}\vec{j}$

is a tangent vector.

Also, $\frac{\vec{v}}{\|\vec{v}\|}$, $-\frac{\vec{v}}{\|\vec{v}\|}$ are both unit tangent vectors.

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}, \quad -\frac{\vec{v}}{\|\vec{v}\|} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$$

$$\therefore \vec{u} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$$

$$\text{or } -\frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$$

(Checking:

$$\nabla f(1,-1) \cdot \frac{\vec{v}}{\|\vec{v}\|} = 0$$

$$= \nabla f(1,-1) \cdot \left(-\frac{\vec{v}}{\|\vec{v}\|}\right)$$
)

(d) & (e) :

Let $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$

where $u_1^2 + u_2^2 = 1$

$$D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \vec{u}$$

$$= 3u_1 - 4u_2$$

(d) $D_{\vec{u}} f(1,-1) = 4$

$\Rightarrow$ $3u_1 - 4u_2 = 4$

$\Rightarrow$ $u_2 = \frac{3}{4} u_1 - 1$

$u_1^2 + u_2^2 = 1$

$\Rightarrow$ $u_1^2 + \left(\frac{3}{4} u_1 - 1\right)^2 = 1$

$\Rightarrow$ $25 u_1^2 - 24 u_1 = 0$

$\Rightarrow$ $u_1 = 0$ or $u_1 = \frac{24}{25}$

If $u_1 = 0$, $u_2 = \frac{3}{4} \cdot 0 - 1 = -1$

If $u_1 = \frac{24}{25}$, $u_2 = \frac{3}{4} \cdot \frac{24}{25} - 1 = -\frac{7}{25}$

$\therefore$ $\vec{u} = -\vec{j}$ or $\vec{u} = \frac{24}{25} \vec{i} - \frac{7}{25} \vec{j}$

(e). $D_{\vec{u}} f(1,-1) = -3$

$\Rightarrow \quad 3u_1 - 4u_2 = -3$

$\Rightarrow \quad u_1 = \frac{4}{3}u_2 - 1$

$u_1^2 + u_2^2 = 1$

$\Rightarrow \quad \left(\frac{4}{3}u_2 - 1\right)^2 + u_2^2 = 1$

$\Rightarrow \quad \frac{25}{9}u_2^2 - \frac{8}{3}u_2 = 0$

$\Rightarrow \quad u_2 = 0 \text{ or } \frac{24}{25}$

If $u_2 = 0$, $u_1 = \frac{4}{3}\cdot 0 - 1 = 0 - 1$

If $u_2 = \frac{24}{25}$, $u_1 = \frac{4}{3}\cdot\frac{24}{25} - 1 = \frac{7}{25}$

$\therefore \vec{u} = -\vec{i}$ or $\vec{u} = \frac{7}{25}\vec{i} + \frac{24}{25}\vec{j}$.